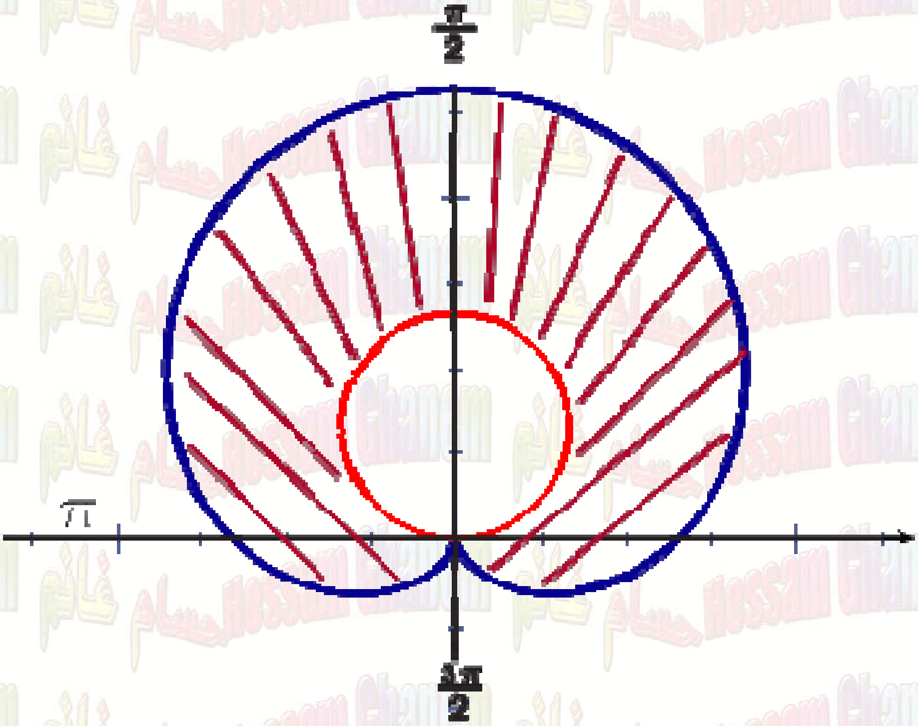
(39) 11.4 Areas In Polar Coordinates

The Area

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

Example 1 Find the area inside the cardioid $r = 2(1 + \sin\theta)$ and outside the circle $r = 2\sin\theta$ — 9 June 1997

Solution



$A = \frac{1}{2}\int_a^b r^2 d\theta$

$A = \frac{1}{2}\int_0^{2\pi} 4(1+\sin\theta)^2\, d\theta - \frac{1}{2}\int_0^{\pi} 4\sin^2\theta\ d\theta$

$= 2\int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta)\, d\theta - 2\int_0^{\pi} \sin^2\theta\ d\theta$

$= 2\int_0^{2\pi} \left(1 + 2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right) d\theta - \int_0^{\pi} (1-\cos 2\theta)\, d\theta$

$= 2\left[\theta - 2\cos\theta + \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} - \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$

$= 2\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} - \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$

$= 2[3\pi - 2 - 0 — 2) - [\pi - 0]$

$= 2(3\pi) - \pi = 5\pi$

$\cos 2\pi = 1$
$\sin 4\pi = 0$
$\sin 2\pi = 0$

Example 2 Find the area of the region outside of $r = 3 + 3\cos\theta$ and inside of $r = 9\cos\theta$ 11 Sep. 1997

Solution

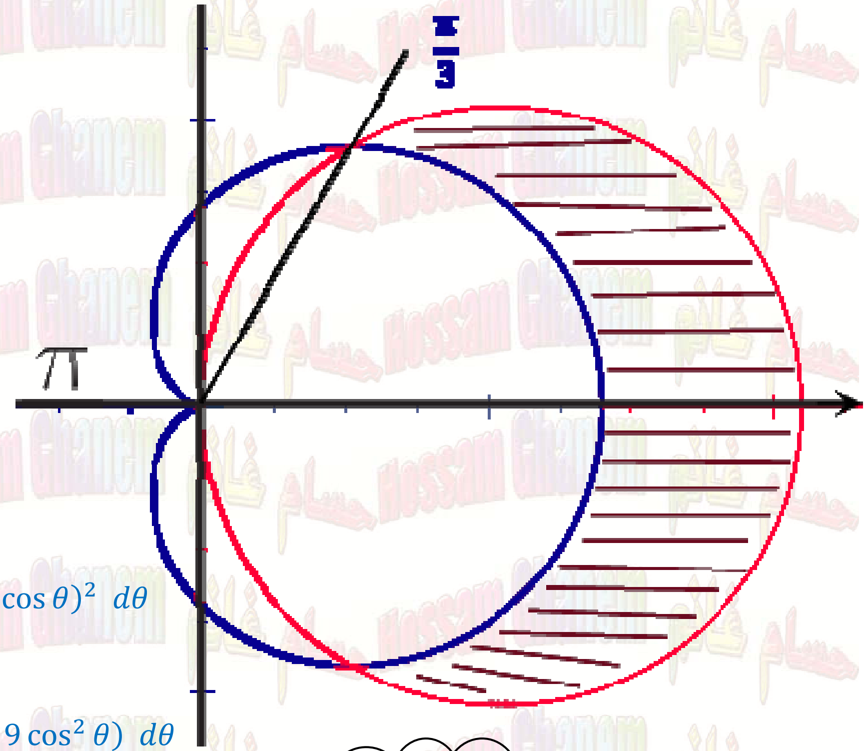
Intersection point

$9\cos\theta = 3 + 3\cos\theta$

$6\cos\theta = 3$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$



$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$A = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} (9\cos\theta)^2\ d\theta\ -\ 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} (3 + 3\cos\theta)^2\ d\theta$$

$$= \int_0^{\frac{\pi}{3}} (81\cos^2\theta)\ d\theta - \int_0^{\frac{\pi}{3}} (9 + 18\cos\theta + 9\cos^2\theta)\ d\theta$$

$$= \int_0^{\frac{\pi}{3}} (72\cos^2\theta - 18\cos\theta - 9)\ d\theta$$

تم جمع التكاملين على بعض لأن حدود التكامل هي نفسها

$$= \int_0^{\frac{\pi}{3}} [36(1 + \cos 2\theta) - 18\cos\theta - 9]\ d\theta$$

$$= \left[\left(36\left(\theta + \frac{1}{2}\sin 2\theta\right) - 18\sin\theta - 9\theta\right)\right]_0^{\frac{\pi}{3}}$$

$$= \left[\left(36\left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - 18\cdot\frac{\sqrt{3}}{2} - 3\right)\right] - 0$$

$$= 12\pi + 9\sqrt{3} - 9\sqrt{3} - 3\pi = 9\pi$$

Example 3 Find the area of the region that is inside the graphs of both equations: $r = 2 + 2\cos\theta$ and $r = 3$. 29 January 2007

Solution

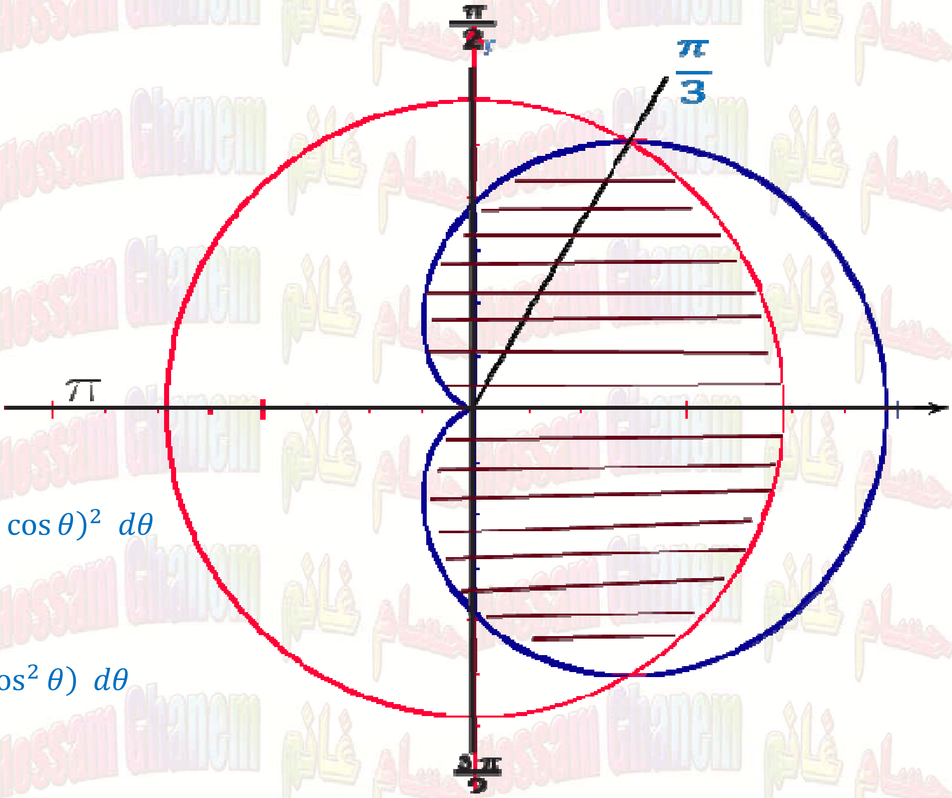
Intersection point

$2 + 2\cos\theta = 3$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$



$$A = \frac{1}{2}\int_a^b r^2\, d\theta$$

$$= 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} (3)^2\ d\theta + 2\cdot\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (2 + 2\cos\theta)^2\ d\theta$$

$$= 9\int_0^{\frac{\pi}{3}} d\theta + \int_{\frac{\pi}{3}}^{\pi} (4 + 8\cos\theta + 4\cos^2\theta)\ d\theta$$

$$= 9\Big[\theta\Big]_0^{\frac{\pi}{3}} + \int_{\frac{\pi}{3}}^{\pi} 4 + 8\cos\theta + 2(1 + \cos 2\theta)\ d\theta$$

$$= 9\left(\frac{\pi}{3} - 0\right) + \left[4\theta + 8\sin\theta + 2\left(\theta + \frac{1}{2}\sin\theta\right)\right]_{\frac{\pi}{3}}^{\pi}$$

$$= 3\pi + [4\pi + 2\pi] - \left[\frac{4\pi}{3} + 4\sqrt{3} + \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right]$$

$$= 3\pi + [6\pi] - \left[\frac{4\pi}{3} + \frac{8\sqrt{3}}{2} + \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right]$$

$$= 3\pi + [6\pi] - \left[\frac{4\pi}{3} + \frac{8\sqrt{3}}{2} + \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right]$$

$$= 9\pi - \left(\frac{6\pi}{3} + \frac{9\sqrt{3}}{2}\right)$$

$$= 7\pi - \frac{9\sqrt{3}}{2}$$

Example 4 Find the area inside the polar curve $r = 1 - \cos\theta$ and outside the polar curve $r = \cos\theta$ 34 August 2009

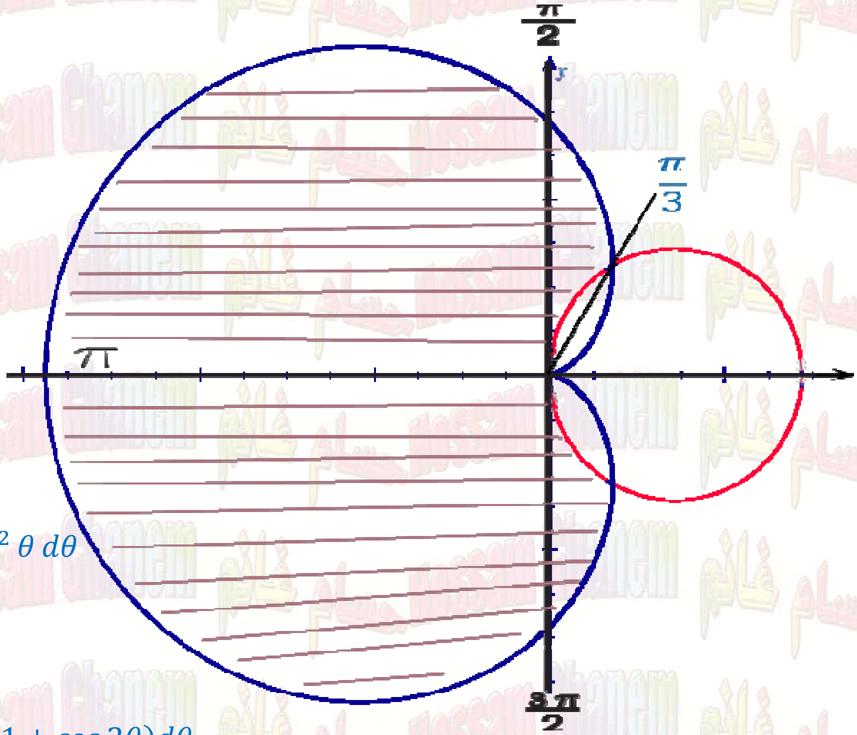
Solution

Intersection point

$$1 - \cos\theta = \frac{1}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$



$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$A = 2\cdot\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}(1-\cos\theta)^2 d\theta - 2\cdot\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta\, d\theta$$

$$A = \int_{\frac{\pi}{3}}^{\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 2\theta)d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi}\left(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta - \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta$$

$$= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi} - \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

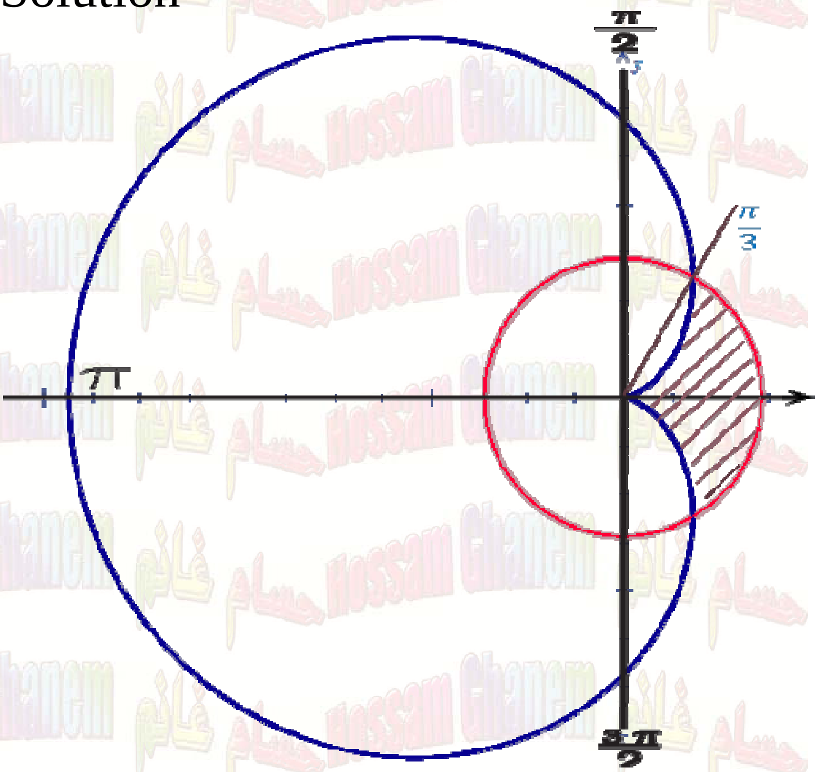
$$= \left[\frac{3\pi}{2} - 0 + 0 - \left(\frac{\pi}{2} - \sqrt{3} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right)\right] - \frac{1}{2}\left[\frac{\pi}{2} + 0 - \left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{3\pi}{2} - \frac{\pi}{2} + \sqrt{3} - \frac{\sqrt{3}}{8} - \frac{\pi}{4} + \frac{\pi}{6} + \frac{\sqrt{3}}{8} = \frac{18 - 6 - 3 + 2}{12}\pi + \sqrt{3} = \frac{11\pi}{12} + \sqrt{3}$$

Example 5 Find the area of the region which lies inside the circle $r=\frac{1}{2}$ and outside the cardioid $r=1-\cos\theta$.

30 January 2008

Solution



Intersection point

$$1-\cos\theta=\frac{1}{2}$$

$$\cos\theta=\frac{1}{2}$$

$$\theta=\frac{\pi}{3}$$

$$A=\frac{1}{2}\int_a^b r^2 d\theta$$

$$A=2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}}\left(\frac{1}{2}\right)^2 d\theta-2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}}(1-\cos\theta)^2 d\theta$$

$$A=\frac{1}{4}\left[\theta\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}(1-2\cos\theta+\cos^2\theta)^2 d\theta$$

$$=\frac{1}{4}\left[\frac{\pi}{3}-0\right]-\int_0^{\frac{\pi}{3}}\left(1-2\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)^2 d\theta$$

$$=\frac{1}{4}\left[\frac{\pi}{3}-0\right]-\int_0^{\frac{\pi}{3}}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)^2 d\theta$$

$$=\frac{\pi}{12}-\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$=\frac{\pi}{12}-\left[\frac{\pi}{2}-\frac{2\sqrt{3}}{2}+\frac{1}{4}\cdot\frac{\sqrt{3}}{2}-0\right]$$

$$=\frac{\pi}{12}-\frac{\pi}{2}+\sqrt{3}-\frac{\sqrt{3}}{8}$$

$$=\frac{\pi}{12}-\frac{6\pi}{12}+\frac{7\sqrt{3}}{8}=\frac{7\sqrt{3}}{8}-\frac{5\pi}{12}$$

Homework

1	Find the area of the region that lies inside the cardioid $r = 4(1 + \sin\theta)$ and outside the circle $r = 4\sin\theta$	13 August 1998
2	Find the area of the region that is inside the circle $r = 9\cos\theta$ and outside the cardioid $r = 3 + 3\cos\theta$.	19 May 2001
3	Sketch the graphs of the polar equations $r = \cos\theta$ and $r = 1 - \cos\theta$, and find the area that lies inside both graphs.	28 July 2006 A
5	Find the area inside the graphs of both polar equations $r = 1 + \cos\theta$ and $r = -\cos\theta$	22 June 2004
6	Find the area inside the graphs of both polar equations $r = 3\cos\theta$ and $r = 1 + \cos\theta$	24 May 2005
7	Find the area of the region that is inside the graph of $r = \sin\theta$ and outside the graph of $r = 1 - \sin\theta$	2 February 1995
8	Find the area of the region that is outside the curve $r = 2 + 2\cos\theta$ and inside the graph of $r = 6\cos\theta$	5 January 1996
9	Sketch in polar coordinates the cardioid $r = 1 + \cos\theta$ and the circle $r = \sin\theta$ and find the area of the region outside the cardioid and inside the circle	6 June 1996
10	Find the area inside both of the polar curves $r = \sin\theta$ and $r = 1 - \sin\theta$	7 August 1996
11	Find the area of the region outside $r = 2(1 + \cos\theta)$ and inside $r = 6\cos\theta$	8 January 1997
12	Show that the polar equation $r = a\sin\theta + b\cos\theta$represents a circle whenever a and b are not both zero, and find its center and radius. (2 points) Find the area of the region that lies in the second quadrant, inside the graph of the polar equation $r = 1 + \cos\theta$, and outside the graph of the polar equation $r = \sin\theta + \cos\theta$ (5 points)	37 August 7, 2010